

2.3 One sided limits and Vertical Asymptotes

Recall in a rational function that is irreducible, we have:

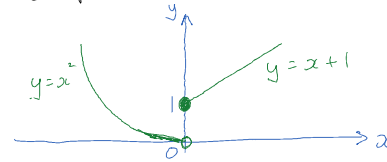
$$y = s(x) + \frac{t(x)}{h(x)}$$

The vertical asymptotes are given by the solutions to $h(x) = 0$

Let us consider a two-piece function

$$f(x) = \begin{cases} x+1, & \text{if } x \geq 0 \\ x^2, & \text{if } x < 0 \end{cases}$$

Its graph is



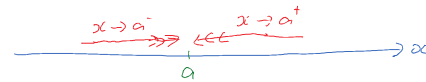
Notice that:-

- ① As x approaches 0 from the left, the function value y approaches 0.
- ② As x approaches 0 from the right, the function value y approaches 1.

How do we write the above statements in mathematics?

Notation:- Suppose we want x to approach a value "a".

- ① If x approaches a from the left, we write $x \rightarrow a^-$
- ② If x approaches a from the right, we write $x \rightarrow a^+$



So back to our example, we write that

- ① $\lim_{x \rightarrow 0^-} f(x) = 0$ $\leftarrow$ we call this the left limit
- ② $\lim_{x \rightarrow 0^+} f(x) = 1$ $\leftarrow$ we call this the right limit

Question:-

What is the overall limit $\lim_{x \rightarrow 0} f(x)$ in the above example?

Definition of Limit

- ① If $\lim_{x \rightarrow a^-} f(x) = L$

$$\text{and } \lim_{x \rightarrow a^+} f(x) = L$$

i.e. $\lim_{x \rightarrow a} f(x) = \lim_{x \rightarrow a^-} f(x) = L$, then

we can define $\lim_{x \rightarrow a} f(x) = L$

② If $\lim_{x \rightarrow a^-} f(x) = M$

and $\lim_{x \rightarrow a^+} f(x) = N$

i.e. $\lim_{x \rightarrow a^-} f(x) \neq \lim_{x \rightarrow a^+} f(x)$,

then we say that $\lim_{x \rightarrow a} f(x)$ does not exist
and is neither $\emptyset$ nor $-\infty$.

OR We say that

$\lim_{x \rightarrow a} f(x)$ is undefined

In our previous example, we have

$$\lim_{x \rightarrow 0^-} f(x) = 0 \quad \text{and} \quad \lim_{x \rightarrow 0^+} f(x) = 1$$

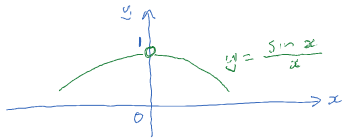
since $\lim_{x \rightarrow 0^-} f(x) \neq \lim_{x \rightarrow 0^+} f(x)$,

therefore

$\lim_{x \rightarrow 0} f(x)$ is undefined.

Another example

Graph of $y = \frac{\sin x}{x}$ near $x = 0$



At $x=0$, $y = \frac{\sin 0}{0} = \frac{0}{0}$ which is undefined.

We can see that

$$\lim_{x \rightarrow 0^-} \frac{\sin x}{x} = 1$$

$$\text{and} \quad \lim_{x \rightarrow 0^+} \frac{\sin x}{x} = 1$$

$$\text{Since} \quad \lim_{x \rightarrow 0^-} \frac{\sin x}{x} = \lim_{x \rightarrow 0^+} \frac{\sin x}{x} = 1$$

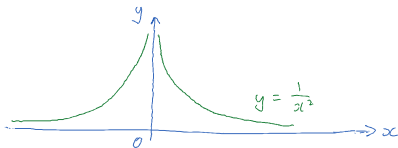
we can define

$$\lim_{x \rightarrow 0} \frac{\sin x}{x} = 1$$

even though $\frac{\sin x}{x}$ is undefined at $x=0$.

Another example:-

Consider $y = \frac{1}{x^2}$



What is $\lim_{x \rightarrow 0} \frac{1}{x^2}$?

$$\lim_{x \rightarrow 0^-} \frac{1}{x^2} = +\infty$$

$$\text{and} \quad \lim_{x \rightarrow 0^+} \frac{1}{x^2} = +\infty$$

$$\text{Since} \quad \lim_{x \rightarrow 0^-} \frac{1}{x^2} = \lim_{x \rightarrow 0^+} \frac{1}{x^2} = +\infty$$

we can define

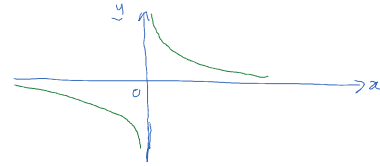
$$\lim_{x \rightarrow 0} \frac{1}{x^2} = +\infty$$

$$\lim_{x \rightarrow 0^-} \frac{1}{x^2} = \frac{1}{\text{+ve no. very close to zero}} = +\infty$$

more to think:-

$$\frac{1}{0.000000000000000001} = 1000000000000000000$$

Consider $y = \frac{1}{x}$



What is $\lim_{x \rightarrow 0} \frac{1}{x}$?

$$\lim_{x \rightarrow 0^-} \frac{1}{x} = \frac{1}{\text{-ve no. very close to zero}} = -\infty$$

$$\text{and } \lim_{x \rightarrow 0^+} \frac{1}{x} = \frac{1}{\text{+ve no. very close to zero}} = +\infty$$

Since $\lim_{x \rightarrow 0^-} \frac{1}{x} \neq \lim_{x \rightarrow 0^+} \frac{1}{x}$, therefore

$\lim_{x \rightarrow 0} \frac{1}{x}$ does not exist[†] and is neither $\emptyset$ nor $\neq \emptyset$.

OR. $\lim_{x \rightarrow 0} \frac{1}{x}$ is undefined.

$$\lim_{x \rightarrow 0^-} \frac{1}{x^2} = +\infty$$

$$\lim_{x \rightarrow 0^+} \frac{1}{x^2} = +\infty$$

$x = 0$ is a
vertical asymptote

$$\lim_{x \rightarrow 0^-} \frac{1}{x} = -\infty$$

$$\lim_{x \rightarrow 0^+} \frac{1}{x} = +\infty$$

$x=0$ is a vertical asymptote

If any one of the following is true :-

$$\textcircled{1} \quad \lim_{x \rightarrow 9^-} f(x) = +\infty$$

(2) $\lim_{x \rightarrow a} f(x) = +\infty$

③ $\lim_{x \rightarrow a^-} f(x) = -\infty$

(4) $\lim_{x \rightarrow a} f(x) = -\infty$

then $x = a$ is
a vertical asymptote
of $y = f(x)$

(12) Does $\lim_{x \rightarrow 2} \frac{x^2 + 12}{(2-x)^2}$ exist?

$$\lim_{x \rightarrow 2^-} \frac{x^2 + 12}{(2-x)^2} = \frac{+ve \text{ no.}}{+ve \text{ no. very close to zero}} = +\infty$$

and $\lim_{x \rightarrow 2^+} \frac{x^2 + 1}{(2 - x)^2} = \frac{+ve \text{ no}}{\text{+ve no. very close to zero}} = +\infty$

Since $\lim_{x \rightarrow 2^-} \frac{x^2+12}{(2-x)^2} = \lim_{x \rightarrow 2^+} \frac{x^2+12}{(2-x)^2} = +\infty$

Therefore

$$\lim_{x \rightarrow 2} \frac{x^2+12}{(2-x)^2} = +\infty$$

And we can also tell that

$x=2$ is a vertical asymptote

for $y = \frac{x^2+12}{(2-x)^2}$

(10) $\lim_{x \rightarrow 4^+} \frac{-3x^3+10x}{\sqrt{x-4}} = \frac{-3(4^3)+10(4)}{\text{+ve no. very close to zero}}$

$$= \frac{\text{-ve no.}}{\text{+ve no. very close to zero}} = -\infty$$

Note

If $a > 0$, then

$\sqrt{a}$ is +ve

and $-\sqrt{a}$ is -ve

Example

$$\sqrt{4} = 2$$

$$-\sqrt{4} = -2$$

2.4 Limits at Infinity and Horizontal Asymptotes

Note:-

If k is any constant and n is a +ve constant, then

$$\lim_{x \rightarrow \infty} \frac{k}{x^n} = 0 \quad \text{and} \quad \lim_{x \rightarrow -\infty} \frac{k}{x^n} = 0$$

How does it happen?

$$\lim_{x \rightarrow -\infty} \frac{k}{x^n} = \frac{k}{(\text{very large -ve no})^n} \approx 0$$

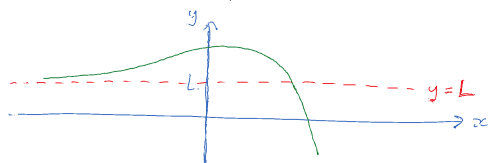
$$\lim_{x \rightarrow +\infty} \frac{k}{x^n} = \frac{k}{(\text{very large +ve no})^n} \approx 0$$

Think about +:-

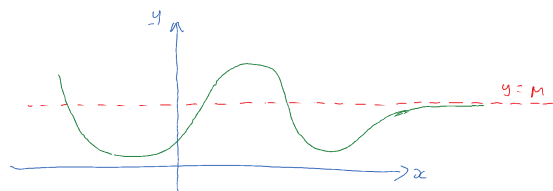
$$\frac{1}{1000000 \dots 0000} \approx 0 \quad \text{and} \quad \frac{1}{-1000000 \dots 0000} \approx 0$$

Generally,

- ① If $\lim_{x \rightarrow -\infty} f(x) = L$, the graph of $y = f(x)$ has a horizontal asymptote $y = L$ to the left of the graph



- ② If $\lim_{x \rightarrow \infty} f(x) = M$, the graph of $y = f(x)$ has a horizontal asymptote $y = M$ to the right of the graph.



Exercises 2.4 page 58

$$\textcircled{1} \lim_{x \rightarrow \infty} \frac{1}{3x^2 + 4x + 1} = \frac{1}{\text{very large +ve no.}} = 0$$

Graph of $y = \frac{1}{3x^2 + 4x + 1}$ has a H.A. to the right.

$$\textcircled{2} \lim_{x \rightarrow -\infty} \frac{2+x}{5x-4} = \frac{-\infty}{-\infty} \text{ which is indeterminate}$$

So we tweak the expression.

$$\lim_{x \rightarrow -\infty} \frac{2+x}{5x-4} = \lim_{x \rightarrow -\infty} \frac{x \left(\frac{2}{x} + 1 \right)}{x \left(5 - \frac{4}{x} \right)}$$

$$= \lim_{x \rightarrow -\infty} \frac{\frac{2}{x} + 1}{5 - \frac{4}{x}}$$

$$\approx \frac{0+1}{5-0} = \frac{1}{5}$$

$$\therefore \lim_{x \rightarrow -\infty} \frac{2+x}{5x-4} = \frac{1}{5}$$

$y = \frac{2+x}{5x-4}$ has a H.A. to the left of the graph.

$$\textcircled{3} \lim_{x \rightarrow -\infty} \frac{2x^2}{x-1} = \lim_{x \rightarrow -\infty} \frac{2x^2}{x \left(1 - \frac{1}{x} \right)}$$

$$= \lim_{x \rightarrow -\infty} \frac{2x}{1 - \frac{1}{x}}$$

$$= \frac{-\infty}{1-0} = -\infty$$

$$\therefore \lim_{x \rightarrow -\infty} \frac{2x^2}{x-1} = -\infty$$

The graph of $y = \frac{2x^2}{x-1}$ has no H.A. to the left.